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Felicitous Labelings of Some Graphs

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1. Introduction :

Graham and Sloane [2] introduced the concept of a harmonious graph. A connected graph G = (V, E) with $|V|$ = p vertices and $|E|$ = q ($\geq$ p) edges is said to be harmonious if it is possible to label the vertices $x \in V$ with distinct numbers f(x) of Z_q, the integers modulo q, in such a way that when each edge e = xy is labeled with f*(e) = (f(x) + f(y)) (mod q), the resulting edge labels are distinct. If the graph is a tree (with p vertices and q = p – 1 edges), it requires exactly one vertex label to be repeated.

To generalize the harmonious labeling and to keep the number of vertex labels that can be repeated to a minimum, S.M. Lee et al [4] intoduced the felicitous labeling. A graph which admits a felicitous labeling is said to be felicitous.

2 : Definitions :

Definition 2.1: A *graph* $G(V,E)$ consists of a finite non-empty set $V = V(G)$ of p *points* (called *vertices*) together with a prescribed set $E = E(G)$ of q unordered pair of distinct vertices of *V*. Each pair $e = \{u, v\}$ of vertices in *E* is a *line* (called *edge*) of *G*.

Definition 2.2: A *vertex switching* G_v of a graph G is obtained by taking a vertex v of G, removing all edges incident to v and adding edges joining v to every vertex not adjacent to v in G.

Definition 2.3: A *vertex switching* of cycle C_n *with one chord* where chord from a triangle with cyclic edges.

Definition 2.4: *Duplication of a vertex* v_k of graph G produces a new graph G_1 by adding a vertex v_k' with $N(v_k') = N(v_k)$. In other words, a vertex v_k' is said to be the duplication of v_k if all the vertices which are adjacent to v_k are now adjacent to v_k' also.

Definition 2.5: For a simple connected graph G, the *square of graph* G is denoted by G^2 and defined as the graph with the same vertex set as of G and two vertices are adjacent in G^2 if they are at a distance 1 or 2 apart in G.

Definition 2.6 [1]: A graph G with q edges is called ***harmonious*** if there is an injection f : V(G) → Z_q, the additive group of integers modulo q such that when each edge xy of G is assigned the label (f(x) + f(y)) (mod q), the resulting edge labels are all distinct. If the graph is a tree (with p nodes and e = p – 1 edges), we require exactly one node label to be repeated.

Definition 2.7 [4]: A ***Felicitous*** labeling of a graph G, with q edges is an injection f : V(G) → {0, 1, 2,. . . , q} so that the induced edge labels $f^*(xy)$ = (f(x) + f(y)) (mod q) are distinct.

3 . Main Results :

Theorem 3.1 : A vertex switching of cycle $(C_n)_v$ is Felicitous for n ≥ 4.

Proof : Let $V((C_n)_v) = \{u\} \cup \{u_i : 1 \le i \le n - 1\}$ and $E((C_n)_v) = \{(u_iu_{i+1}) : 1 \le i \le n - 2\} \cup \{(uu_i) : 2 \le i \le n - 2\}$.

Case (i) : when n ø 1 (mod 2)

Define a function f : $V((C_n)_v) \rightarrow \{0,1,2, . . . , q = 2n - 5\}$ by

$$f(u) = \frac{3n-7}{2}$$

$$f(u_i) = \begin{cases} \frac{i-1}{2}, & 1 \le i \le n - 2 \text{ and } i ø 1 \ (\text{mod } 2) \\ \frac{n+i-3}{2}, & 2 \le i \le n - 1 \text{ and } i ø 0 \ (\text{mod } 2) \end{cases}$$

The edge labels are as follows :

$$f_1(u_i\, u_{i+1}) = \frac{n-1}{2} + i - 1, \quad 1 \le i \le n - 2$$

$$f_1(uu_i) = \begin{cases} 2n - 5 + \frac{i}{2}, & 2 \le i \le n - 3 \text{ and } i ø 0 \ (\text{mod } 2) \\ \frac{3n+i-3}{2}, & 3 \le i \le n - 2 \text{ and } i \equiv 1 \ (\text{mod } 2) \end{cases}$$

Clearly, $f_1((C_n)_v) = \{\frac{n-1}{2}, \frac{n-1}{2} + 1, \ . . . \ \frac{n-1}{2} + n - 3\} \cup \{\frac{3n+3-8}{2}, \frac{3n+5-8}{2}, \ . . . ,$
$\frac{3n+n-2-8}{2}\} \cup \{2n - 5 + 1, 2n - 5 + 2, \ . . . \ , 2n - 5 + \frac{n-3}{2}\}$

$= \{\frac{n-1}{2}, \frac{n-3}{2}, . . . , \frac{3n-7}{2}\} \cup \{\frac{3n-5}{2}, \frac{3n-3}{2}, . . . , 2n - 5\} \cup$
$\{2n - 4, 2n - 3, \ . . . , \frac{5n-13}{2}\}$

$= \{\frac{n-1}{2}, \frac{n-3}{2}, . . . , \frac{3n-7}{2}, \frac{3n-5}{2}, \frac{3n-3}{2}, . . . , 2n - 5, 2n - 4, 2n - 3,$
$. . . , \frac{5n-13}{2}\}$

Then, $f^*(E((C_n)_v))$ = $f_1(E((C_n)_v)) \pmod{2n-5}$

= $\{\frac{n-1}{2}, \frac{n-3}{2}, \ldots, \frac{3n-7}{2}, \frac{3n-5}{2}, \frac{3n-3}{2}, \ldots, 2n-5,$

$2n-4, 2n-3, \ldots, \frac{n-3}{2}\}$

Hence, $((C_n)_v)$ is a felicitous graph for $n \geq 4$.

For example, a felicitous labeling of $((C_7)_v)$, $n \equiv 1 \pmod 2$ is shown in Fig. 3.1.

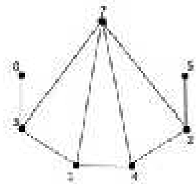

Fig. 3.1

Case (ii) : when n ø 0 (mod 2)

Define a function $f : V((C_n)_v) \rightarrow \{0,1,2, \ldots, q = 2n-5\}$ by

$$f(u) = \frac{3n-6}{2}$$

$$f(u_i) = \begin{cases} \frac{3n-8}{2}, & i = 1 \\ \frac{i-2}{2}, & 2 \leq i \leq n-2 \text{ and } i \text{ ø } 0 \pmod 2 \\ \frac{n+i-5}{2}, & 3 \leq i \leq n-1 \text{ and } i \equiv 1 \pmod 2 \end{cases}$$

The edge labels are as follows :

$$f_1(u_i u_{i+1}) = \frac{n-6}{2} + i, 2 \leq i \leq n-2$$

$$f_1(u_1 u_2) = \frac{3n-8}{2}$$

$$f_1(uu_i) = \begin{cases} \frac{3n+i-8}{2}, & 2 \leq i \leq n-2 \text{ and } i \text{ ø } 0 \pmod 2 \\ \frac{4n+i-11}{2}, & 3 \leq i \leq n-3 \text{ and } i \text{ ø } 1 \pmod 2 \end{cases}$$

Clearly, $f_1((C_n)_v)$ = $\{\frac{n-6}{2} + 2, \frac{n-6}{2} + 3, \ldots \frac{n-6}{2} + n - 2\} \cup \{\frac{3n-8}{2}\}$

$\cup \{\frac{3n+2-8}{2}, \frac{3n+4-8}{2}, \ldots, \frac{3n+n-2-8}{2}\} \cup \{\frac{4n+3-11}{2}, \frac{4n+5-11}{2}, \ldots, \frac{4n+n-3-11}{2}\}$

= $\{\frac{n-2}{2}, \frac{n}{2}, \ldots, \frac{3n-10}{2}\} \cup \{\frac{3n-8}{2}\} \cup \{\frac{3n-6}{2}, \frac{3n-4}{2}$

$, \ldots, 2n-5\} \cup \{2n-4, 2n-3, \ldots, \frac{5n-14}{2}\}$

= $\{\frac{n-2}{2}, \frac{n}{2}, \ldots, \frac{3n-10}{2}, \frac{3n-8}{2}, \frac{3n-6}{2}, \ldots, 2n-5, 2n$

$-4, 2n-3, \ldots, \frac{5n-14}{2}\}$

Then, $f^*(E((C_n)_v))$ = $f_1(E((C_n)_v)) \pmod{2n-5}$

= $\{\frac{n-2}{2}, \frac{n}{2}, \ldots, \frac{3n-10}{2}, \frac{3n-8}{2}, \frac{3n-6}{2}, \ldots, 2n-5,$

$2n-4, 2n-3, \ldots, \frac{n-4}{2}\}$

Hence, $((C_n)_v)$ is a Felicitous graph for $n \geq 4$.

For example, a Felicitous labeling of $((C_7)_v)$, $n \equiv 0 \pmod 2$ is shown in Fig. 3.2.

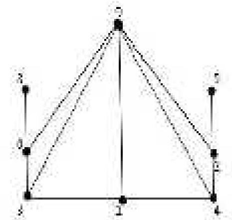

Fig. 3.2

Theorem 3.2 : A vertex switching of cycle C_n, $n \geq 4$ with one chord is Felicitous.

Proof : Let G be the vertex switching of cycle with one chord.

Let $V(G) = \{u\} \cup \{u_i : 1 \leq i \leq n - 1\}$ and $E(G) = \{(u_iu_{i+1}) : 1 \leq i \leq n - 2\} \cup \{(u_1u_{n-1})\} \cup \{(uu_i) : 2 \leq i \leq n - 2\}$.

Case (i) : when n ø 1 (mod 2)

Define a function $f : V(G) \rightarrow \{0,1,2, \ldots, q = 2n - 4\}$ by

$$f(u) = 2n - 4$$

$$f(u_i) = \begin{cases} \frac{n-4+i}{2}, & 1 \leq i \leq n - 4 \text{ and } i \text{ ø } 1 \pmod 2 \\ (n-1) + \frac{i-2}{2}, & 2 \leq i \leq n - 3 \text{ and } i \text{ ø } 0 \pmod 2 \\ n - 2, & i = n - 2 \\ 2n - 5, & i = n - 1 \end{cases}$$

The edge labels are as follows :

$$f_1(u_i\, u_{i+1}) = \begin{cases} \frac{3n-7}{2} + i, & 1 \leq i \leq n - 4 \\ \frac{5n-11}{2}, & i = n - 3 \\ 3n - 7, & i = n - 2 \end{cases}$$

$$f_1(u_1\, u_{n-1}) = \frac{5n-13}{2}$$

$$f_1(uu_i) = \begin{cases} \frac{5n-12+i}{2}, & 3 \leq i \leq n - 4 \text{ and } i \text{ ø } 1 \pmod 2 \\ (3n-5) + \frac{i-2}{2}, & 3 \leq i \leq n - 3 \text{ and } i \text{ ø } 0 \pmod 2 \\ 3n - 6, & i = n - 2 \end{cases}$$

Clearly, $f_1(G)$ = $\{\frac{3n-7}{2}+1, \frac{3n-7}{2}+2, \ldots, \frac{3n-7}{2}+n-4\} \cup \{\frac{5n-13}{2}\} \cup \{\frac{5n-11}{2}\} \cup \{\frac{5n-12+3}{2}, \frac{5n-12+5}{2}, \ldots, \frac{5n-12+n-4}{2}\} \cup \{3n-7\} \cup \{3n-6\} \cup \{3n-5, 3n-5+1, \ldots, 3n-5+\frac{n-1}{2}\}$

= $\{\frac{3n-5}{2}, \frac{3n-3}{2}, \ldots, \frac{5n-15}{2}\} \cup \{\frac{5n-13}{2}\} \cup \{\frac{5n-11}{2}\} \cup \{\frac{5n-9}{2}, \frac{5n-7}{2}, \ldots, 3n-8\} \cup \{3n-7\} \cup \{3n-6\} \cup \{3n-5, 3n-4, \ldots, \frac{7n-11}{2}\}$

= $\{\frac{3n-5}{2}, \frac{3n-3}{2}, \ldots, \frac{5n-15}{2}, \frac{5n-13}{2}, \frac{5n-11}{2}, \frac{5n-9}{2}, \frac{5n-7}{2}, \ldots, 3n-8, 3n-7, 3n-6, \ldots, \frac{7n-11}{2}\}$

Then, $f^*(E(G))$ = $f_1(E(G)) \pmod{2n-5}$

= $\{\frac{3n-5}{2}, \frac{3n-3}{2}, \ldots, \frac{5n-15}{2}, \frac{5n-13}{2}, \frac{5n-11}{2}, \frac{5n-9}{2}, \frac{5n-7}{2}, \ldots, 3n-8, 3n-7, 3n-6, \ldots, \frac{3n-3}{2}\}$

Hence, a vertex switching of cycle C_n, $n \geq 4$ with one chord is a Felicitous graph.

For example, a Felicitous labeling of a vertex switching of cycle C_9 with one chord is shown in Fig. 3.3.

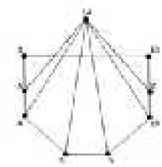

Fig. 3.3

Theorem 3.3 : Vertex duplication of C_n is Felicitous.

Proof : Let G be the graph of vertex duplication of cycle C_n, $n \geq 3$.

Let $V(G) = \{u\} \cup \{u_i : 1 \leq i \leq n\}$ and $E(G) = \{(u_i u_{i+1}) : 1 \leq i \leq n-1\} \cup \{(u_1 u_n)\} \cup \{(u u_2)\} \cup \{(u u_n)\}$.

Case (i) :when n ø 1, 3 (mod 4)

Define a function $f : V(G) \rightarrow \{0,1,2, \ldots, q = n+2\}$ by

$$f(u) = 0$$

$$f(u_i) = \begin{cases} \frac{i+1}{2}, & 1 \leq i \leq n \text{ and } i ø 1 \pmod 2 \\ \frac{n+3+i}{2}, & 2 \leq i \leq n-1 \text{ and } i ø 0 \pmod 2 \end{cases}$$

The edge labels are as follows :

$$f_1(u_i u_{i+1}) = \frac{n+5+i}{2}, \quad 1 \leq i \leq n-1$$

$$f_1(u_1 u_n) = \frac{n+3}{2}$$

$f_1(u\,u_2) = \frac{n+5}{2}$

$f_1(u\,u_n) = \frac{n+1}{2}$

Clearly, $f_1(G) = \{\frac{n+1}{2}\} \cup \{\frac{n+3}{2}\} \cup \{\frac{n+5}{2}\} \cup \{\frac{n+5}{2}+1, \frac{n+5}{2}+2, \ldots, \frac{n+5}{2}+n-1\}$

$= \{\frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, \frac{3n+3}{2}\}.$

Then, $f^*(E(G)) = f_1(E(G)) \pmod{n+2}$

$= \{\frac{n+1}{2}, \frac{n+3}{2}, \frac{n+5}{2}, \ldots, \frac{n-1}{2}\}.$

Hence, vertex duplication of C_n, $n \equiv 1,3$ (mode 4) is Felicitous.

For example, a Felicitous labeling of a vertex duplication of cycle C_7 and C_9 is shown in Fig. 3.4.

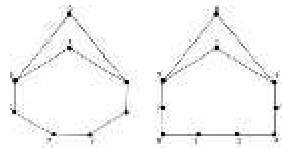

Fig. 3.4

Case (ii) : when n ø 0 (mod 4)

Define a function $f : V(G) \rightarrow \{0,1,2,\ldots, q = n+2\}$ by

$f(u) = 0$

$$f(u_i) = \begin{cases} \frac{n}{2}, & i = 1 \\ \frac{i-1}{2}, & 3 \le i \le n-1 \text{ and } i \text{ ø } 1 \pmod 2 \\ \frac{n+i}{2}, & 2 \le i \le \frac{n}{2} \text{ and } i \text{ ø } 0 \pmod 2 \\ \frac{n+i}{2}+2, & \frac{n}{2}+1 \le i \le n \text{ and } i \text{ ø } 0 \pmod 2 \end{cases}$$

The edge labels are as follows :

$$f_1(u_iu_{i+1}) = \begin{cases} n+1, & i = 1, \\ \frac{n+2i}{2}, & 2 \le i \le \frac{n}{2} \\ \frac{n}{2}+i+2, & \frac{n}{2}+1 \le i \le n-1 \end{cases}$$

$f_1(u_1\,u_n) = \frac{3n+4}{2}$

$f_1(u\,u_2) = \frac{n+2}{2}$

$f_1(u\,u_n) = n+2$

Clearly, $f_1(G) = \{\frac{n+2}{2}\} \cup \{\frac{n+4}{2}, \frac{n+6}{2}, \ldots, n\} \cup \{n+1\} \cup \{n+2\} \cup \{\frac{n}{2}+\frac{n}{2}+3, \frac{n}{2}+\frac{n}{2}+4, \ldots, \frac{n}{2}+n-1+2\} \cup \{\frac{3n+4}{2}\}.$

$= \{\frac{n+2}{2}, \frac{n+4}{2}, \frac{n+6}{2}, \ldots, n, n+1, n+2, \ldots, \frac{3n+2}{2}, \frac{3n+4}{2}\}.$

Then, $f^*(E(G)) = f_1(E(G)) \pmod{n+2}$

$= \{\frac{n+2}{2}, \frac{n+4}{2}, \frac{n+6}{2}, \ldots, n, n+1, \ldots, \frac{n}{2}\}.$

$= \{\frac{n+2}{2}, \frac{n+4}{2}, \frac{n+6}{2}, \ldots, \frac{n}{2}\}.$

Hence, vertex duplication of C_n, $n \equiv 0$ (mode 4) is Felicitous.

For example, a Felicitous labeling of a vertex duplication of cycle C_8 is shown in Fig. 3.5.

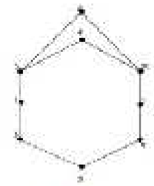

Fig. 3.5

Theorem 3.4 : $B^2_{n,n}$ is a felicitous graph for $n \geq 2$.

Proof : Let $V(B^2_{n,n}) = \{u_o, v_o\} \cup \{u_i, v_i : 1 \leq i \leq n\}$ and $E(B^2_{n,n}) = \{(u_o v_o)\} \cup \{(u_o u_i) \cup (v_o v_i) \cup (v_o u_i) \cup (u_o v_i) : 1 \leq i \leq n\}$.

Define a function $f : V(B^2_{n,n}) \rightarrow \{0,1,2,\ldots, q = 4n+1\}$ by

$f(u_o) = n$, $f(v_o) = 2n$

$f(u_i) = i$, $1 \leq i \leq n$

$f(v_i) = 2n + i$, $1 \leq i \leq n$

The edge labels are as follows :

$f_1(u_o u_i) = n + i - 1$, $1 \leq i \leq n$

$f_1(v_o u_i) = 2n + i - 1$, $1 \leq i \leq n$

$f_1(u_o v_o) = 3n$,

$f_1(u_o v_i) = 3n + i$, $1 \leq i \leq n$

$f_1(v_o v_i) = 4n + i$, $1 \leq i \leq n$

Clearly, $f_1(E(B^2_{n,n})) = \{n, n+1, \ldots, 2n-1\} \cup \{2n, 2n+1, \ldots, 3n-1\} \cup \{3n\} \cup \{3n+1, 3n+2, \ldots, 4n\} \cup \{4n+1, 4n+2, \ldots, 5n\}$

$= \{n, n+1, \ldots, 2n-1, 2n, 2n+1, \ldots, 3n-1, 3n, 3n+1, 3n+2, \ldots, 4n, 4n+1, 4n+2, \ldots, 5n\}$

Then, $f^*(E(B^2_{n,n})) = f_1(E(B^2_{n,n})) \pmod{4n+1}$

$= \{n, n+1, \ldots, 2n-1, 2n, 2n+1, \ldots, 3n-1, 3n, 3n+1, 3n+2, \ldots, 4n, 4n+1, 1, 2, \ldots, n-1\}.$

$= \{1, 2, \ldots, n-1, n, n+1, \ldots, 4n, 4n+1\}.$

Hence, $B_{n,n}^{2}$ is a felicitous graph for n ≥ 2.

For example, a felicitous labeling of $B_{4,4}^{2}$ is shown in Fig. 3.6.

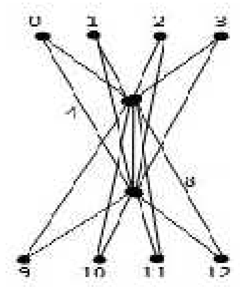

Fig.3.6

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